Review of Introduction to Probability and Statistics

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Homework #1 Solutions

1. Define the sample space (that is, list all of the elements) for each example below:
   a. the set of integers between 1 and 60 evenly divisible by 7
   b. the set $\Omega = \{x \mid x^2 + 4x - 5 = 0\}$
   c. the set of outcomes when a coin is tossed until a tail or three heads appear
   d. all points in the first quadrant inside a circle of radius 3 with center at the origin

Solution:

(a) $\Omega = \{7, 14, 21, 28, 35, 42, 49, 56\}$.

(b) For $x^2 + 4x - 5 = (x + 5)(x - 1) = 0$, the solutions are $x = -5$ and $x = 1$. So, $\Omega = \{-5, 1\}$.

(c) Ω = {T,HT,HHT,HHH}.

(d) $\Omega = \{(x, y) \mid x^2 + y^2 < 9; x \geq 0; y \geq 0\}$.

2. An engineer tests three products and pronounces the results for each as pass (P) or fail (F).
   a. List the elements of a sample space Ω
   b. List the elements corresponding to the event E that at least two products pass
   c. Describe the event that has as its elements {FFF, PFF, FFP, PFP}

Solution:

(a) Ω = {FFF, FFP, FPF,PFF, FPP, PFP, PPF, PPP}.

(b) E = {FFF, FFP, FPF,PFF}.

(c) The second product passed the test.

3. In a battery life study, each of 3 laptops is tested using 5 different brands of batteries running 7 different software suites. If 2 testers are used in the study, and test runs are made once under each distinct set of conditions for each tester, how many test runs are needed?

Solution:

With $n_1 = 3$ laptops, $n_2 = 5$ brands of batteries, $n_3 = 7$ software suites, and $n_4 = 2$ testers,

the product rule yields (3)(5)(7)(2) = 210 test runs.

4. A recent study concluded that by following seven basic health rules a man's life can be extended by 11 years on the average and a woman's life by 7 years. These seven rules are: don't smoke, exercise regularly, use alcohol moderately, get 7 to 8 hours of sleep each night, maintain proper weight, eat breakfast every day, and do not eat between meals. In how many ways can a person adopt five of these rules:
   a. If the person presently violates all 7 rules?
   b. If the person never drinks and always eats breakfast?

Solution:

Note that I am choosing without replacement, and that order doesn't matter.

(a) There are $\binom{7}{5}$ = 21 combinations.
(b) Since two of the choices are fixed, I am now choosing three out of the five that remain: $\binom{5}{3}$ = 10 combinations.

5. How many permutations can be made from the letters of the word *healthy*? How many of these permutations start with the letter y?

Solution:

For seven unique letters there are 7! = 5040 permutations. In the second question, since the first letter must be y, the remaining 6 letters can be arranged in 6! = 720 ways.

6. How many unique permutations can be made from the letters of the word *infinity*?

Solution:

Consider first the case where each letter i and each letter n are unique (a different color for each repeated letter, for example). Then, there would be 8! permutations. Consider now just the orderings of the letters i. Since there are three, there are 3! ways to order the differently colored i's. Since these orderings are all the same, that leaves 8!/3! unique arrangements. Similarly, there are 2! orderings for the two letter n's. This gives 8!/(3!2!) = 3360 unique permutations.

7. Given the digits 0, 1, 2, 3, 4, 5, and 6,
   a. How many three-digit numbers can be formed from if each digit can be used only once?
   b. How many of these are odd numbers?
   c. How many of these are greater than 330?

d. How many three-digit numbers can be formed from if each digit can be used more than once?

Solution:

(a) Any of the 6 nonzero digits can be chosen for the hundreds position, leaving 6 digits possible for the tens position, and 5 digits for the units position. So, there are (6)(6)(5) = 180 three digit numbers.

(b) The units position can be filled using any of the 3 odd digits. Any of the remaining 5 nonzero digits can be chosen for the hundreds position, leaving a choice of 5 digits for the tens position. Thus, there are (3)(5)(5) = 75 three digit odd numbers. Note that this is half of the answer to (a), as one might expect.

(c) If a 4, 5, or 6 is used in the hundreds position there remain 6 and 5 choices, respectively, for the tens and units positions. This gives (3)(6)(5) = 90 three digit numbers beginning with a 4, 5, or 6. If a 3 is used in the hundreds position, then a 4, 5, or 6 must be used in the tens position leaving 5 choices for the units position. In this case, there are (1)(3)(5) = 15 three digit number begin with a 3. So, the total number of three digit numbers that are greater than 330 is 90 + 15 = 105.

(d) Selecting with replacement, and keeping in mind that the first digit cannot be zero, there are (6)(7)(7) = 294 three digit numbers possible.

8. In how many ways can 5 people be seated around a circular table?

Solution:

Since there is no "start" to a circular sequence, begin by placing one person anywhere. Then, there are 4! =24 permutations for the remaining four people. Alternately, consider first a straight line of people, with 5! permutations. If I sit this line of people around the table (keeping their order), there are five ways to do that, all equivalent since there is no "start" to the circle. Thus, the number of unique seatings is 5!/5 = 4!.

9. In wedding photo, how many ways can I line up the bride, groom, and 5 other guests making sure that the bride and groom stand next to each other?

Solution:

Consider the following two stages. Frist, line up the bride and the five guests. There are 6! permutations. Next, insert the groom into the line. There are two choices: to the right or left of the bride. This gives (6!)(2) = 1440 permutations. Note that the key to making this problem easy is to break it down into stages with known counts.

10. Genes can be thought of as paragraphs comprised of three letter words, where the letters are A, T, G or C. Each word or "codon" calls out an amino acid of the protein the gene codes for. For example, GGG is a valid codon for the amino acid glycine. How many different codon words can be written with the four letters if the letters can be repeated?

Solution:

Applying the stage counting method, there are four choices for each letter, three letters, so there are $4^3 = 64$ codon words. (It turns out that 61 codons specify the amino acids used in proteins (some of them are redundant) and 3 codons (called stop codons) are used to signal termination of growth of the polypeptide chain.)

11. Consider a discrete sample space $\Omega = \{1, 2, 3, 4\}$. Determine if the following probability laws are valid.

    a. $P(1) = 0.26$, $P(2) = 0.25$, $P(3) = 0.26$, $P(4) = 0.25$.

    b. $P(1) = 0.15$, $P(2) = 0.28$, $P(3) = 0.33$, $P(4) = 0.24$.

    c. $P(1) = 0.26$, $P(2) = 0.35$, $P(3) = -0.04$, $P(4) = 0.43$.

Solution:

(a) No. $P(\Omega) = P(1) + P(2) + P(3) + P(4)$ must equal 1.

(b) Yes.

(c) No. Negative probabilities are not allowed.

12. One-third of Americans are overweight. Selecting two Americans at random, what is the chance that both are overweight?

Solution:

A = an American is overweight. $P(A) = 1/3$ (by the law of large numbers). Selecting two Americans at random, we can assume that the two events (people) are independent. Let A_1 and A_2 be the events that first and second person are overweight, respectively. We want $P(A_1 \cap A_2)$. Since the events are independent, $P(A_1 \cap A_2) = P(A_1)\, P(A_2) = 1/9$.

13. A batch of one hundred widgets is inspected by testing four widgets selected at random. The batch will be rejected if one or more widgets from the sample is found to be defective. What is the probability that the batch will be accepted if the batch has 5 defective widgets?

Solution:

Approach 1, using conditional probabilities. Let A_i be the event that the i^{th} sample is not defective. We can think of the sampling as beginning with taking the first item out of the full population. If that one is not defective, then we take a sample from the remaining 99. We can write the probability that the batch will be accepted as

$$P(A_1)\,P(A_2|A_1)\,P(A_3|A_1 \cap A_2)\,P(A_4|A_1 \cap A_2 \cap A_3) = \frac{95}{100}\frac{94}{99}\frac{93}{98}\frac{92}{97} = 0.812$$

Approach 2, using counting. Count how many ways there are of drawing 4 good devices from the 95 good devices in the batch, then divide by the total number of ways of drawing any 4 devices from the batch of 100:

$$\frac{\binom{95}{4}}{\binom{100}{4}} = \frac{\frac{95!}{4!\,91!}}{\frac{100!}{4!\,96!}} = \frac{95\cdot 94\cdot 93\cdot 92}{100\cdot 99\cdot 98\cdot 97} = 0.812$$

14. Consider an experiment where two fair 6-sided dice (marked with numbers 1 through 6) are thrown and all 36 possible outcomes are equally likely. Are the events A = {first roll is a 1} and B = {sum of the two rolls is 7} independent?

Solution:

To prove independence, we must show that $P(A \cap B) = P(A)\,P(B)$. So let's calculate each probability.

There is only one way for A and B to be true: the dice rolled (1,6). The probability of this happening is 1/36. Thus, $P(A \cap B) = 1/36$.

The probability of rolling a 1 with the first die is 1/6. Thus, $P(A) = 1/6$.

The ways of getting a sum of 7 are (1,6), (2,5), (3,4), (4,3), (5,2), and (6,1). Thus, $P(B) = 6/36 = 1/6$.

Multiplying $P(A)\,P(B)$ we find that result is 1/36, the same as $P(A \cap B)$. Thus, the two events are independent.

Interestingly, the independence comes about because getting a sum of 7 allows all six possible values for the roll of the first die, each equally likely.

15. You are playing in a single-elimination scrabble tournament (one loss and you lose the tournament). The odds makers say the probability of you beating half the players (call them type 1 players) is 0.3, the probability of beating a quarter of the players (type 2 players) is 0.4, and the probability of beating the remaining quarter (type 3 players) is 0.5. What is the probability of winning any random game?

Solution:

Define A_i = the event of playing against a type i player. The problem says that

$P(A_1) = 0.5$, $P(A_2) = 0.25$, $P(A_3) = 0.25$

Let B = winning a game. Thus, the problem tells us that

$P(B|A_1) = 0.3$, $P(B|A_2) = 0.4$, $P(B|A_3) = 0.5$

Now, apply the total probability theorem. The probability of winning is

$P(B) = P(A_1)\, P(B|A_1) + P(A_2)\, P(B|A_2) + P(A_3)\, P(B|A_3) = (0.5)(0.3) + (0.25)(0.4) + (0.25)(0.5)$

$P(B) = 0.375$

16. Referring to the tournament of problem 15, suppose that you won the first game. What is the probability that the first game was played against a type 1 player?

Solution:

We want $P(A_1|B)$. Since we are given $P(B|A_1)$, we can flip the conditional probability with Bayes' Theorem.

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(B)} = \frac{(0.3)(0.5)}{0.375} = 0.4$$